On partially ordered algebras. II

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Dedicated to L. Kalmár on his 60th birthday

This note is the continuation of the author's paper "On partially ordered algebras. I" to appear in *Colloquium Mathematicum.* It will be shown that every algebra gives rise to a lattice-ordered algebra where the monotony domains coincide with the whole underlying set and all the operations are isotone in each of their variables.[1]) Some results on ideal systems can be generalized mutatis mutandis to our more general case.

1. Lattice-ordered algebras obtained from arbitrary algebras

We are going to show how from every algebra it is possible to derive a lattice-ordered algebra e. g. as the set of subalgebras or subsystems of a special kind. This is the way in which the lattice of subgroups, normal subgroups of groups, subrings, left ideals or ideals of rings, l-ideals of lattice-ordered groups, submodules of modules etc. are obtained.

In order to frame our definition more generally, we recall the concept of ideal systems. We formulate this here in a way which suites better to our purposes.

Let $(A; F)$ be an arbitrary algebra and

$$X \to X_r$$

a mapping from the set of all non-void finite subsets of A into the set of all subsets of A satisfying the following conditions:

1°. $X \subseteqq X_r$,

2°. $X \subseteqq Y_r$ implies $X_r \subseteqq Y_r$.

(That is to say, it satisfies the axioms of closure operators.) Usually, $(a)_r$ is prescribed in some way.

We extend the domain of definition of this r-operation to infinite subsets Z of A by setting [2])

3°. $Z_r = \cup X_r$ where X runs over all finite subsets of Z.

Then, as readily checked, axioms 1°—2° hold for infinite X, Y as well. The system

[1]) For the basic definitions on partially ordered algebras in general we refer to Part I. For partially ordered groups and rings see e. g. [1].

[2]) We shall use the signs $\wedge$, $\vee$ to denote lattice-operations, while $\cap$, $\cup$ will denote set-theoretical meet and join, respectively.

of all X_r with finite and infinite X will be denoted by A_r and called the *r-ideal system* of A.

A_r is partially ordered under inclusion. It is closed with respect to intersection — this is an immediate consequence of 1° and 2°. If we define

$$X_r \vee Y_r = (X \cup Y)_r$$

which is easily shown to be independent of the representation of X_r and Y_r by X and Y, respectively), then A_r becomes a lattice. In A_r we have evidently

(1) $$\bigvee_\alpha X_r^\alpha = (\bigcup_\alpha X^\alpha)_r$$

with α running over an arbitrary index set.

For each $f \in F$ we set[3])

$$\bar{f}(X_r^1, \ldots, X_r^n) = (\cup f(x^1, \ldots, x^n) \quad \text{for all} \quad x^i \in X_r^i)_r;$$

then every $\bar{f}$ in A_r satisfies the monotony law with the whole of A_r as monotony domain. The operations $\bar{f}$ are evidently isotone in their variables. Hence A_r becomes a lattice-ordered algebra with the operations $\bar{f}$ $(f \in F)$ and $\wedge, \vee$; moreover A_r is a complete lattice under $\wedge, \vee$. However, not all the identities of A retain their validity in A_r.

Call an operation $f \in F$ *r-admissible* if for each i it satisfies

(2) $$(f(x^1, \ldots, x^{i-1}, X_r^i, x^{i+1}, \ldots, x^n))_r =$$
$$= (f(x^1, \ldots, x^{i-1}, x^i, x^{i+1}, \ldots, x^n) \quad \text{for all} \quad x^i \in X^i)_r.$$

Recall that in partially ordered groups or in commutative rings only r-ideal systems are considered which are supposed to have the property $(aX)_r = aX_r$ which is somewhat stronger than (2).

For r-admissible operations $f \in F$ we have:

(3) $$\bar{f}(X_r^1, \ldots, X_r^n) = (f(X^1, \ldots, X^n))_r.$$

In fact, we have on using (1)

$$(f(X_r^1, X^2, \ldots, X^n))_r = (\cup f(X_r^1, x^2, \ldots, x^n) \quad \text{for all} \quad x^i \in X^i, i \geqq 2)_r =$$
$$= \vee (f(X_r^1, x^2, \ldots, x^n))_r =$$
$$= \vee [\vee (f(x^1, x^2, \ldots, x^n) \quad \text{for} \quad x^1 \in X^1)_r \quad \text{for} \quad x^i \in X^i, i \geqq 2]_r =$$
$$= \vee (f(x^1, x^2, \ldots, x^n) \quad \text{for} \quad x^i \in X^i)_r = (f(X^1, X^2, \ldots, X^n))_r$$

whence the trivial conclusion is derived that, in $(f(X^1, \ldots, X^n))_r$, X^1 can be

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[3]) We could write the definition of $\bar{f}$ as

$$\bar{f}(X_r^1, \ldots, X_r^n) = (f(X_r^1, \ldots, X_r^n))_r$$

if we mean by $f(Y^1, \ldots, Y^n)$ for subsets Y^i of A the set of all $f(y^1, \ldots, y^n)$ with $y^i \in Y^i$. This notation will be used below.

replaced by its closure X_r^1, and clearly the same holds for each $i=2, \ldots, n$. This establishes (3).

Next we verify the following two assertions concerning r-admissible operations.

Theorem 1. *If f is an r-admissible operation, then $\bar{f}$ is infinitely distributive over $\vee$.*

We have evidently

$$\bar{f}(\bigvee_{\alpha} X_r^{\alpha}, X_r^2, \ldots, X_r^n) = \bar{f}((\bigcup_{\alpha} X^{\alpha})_r, X_r^2, \ldots, X_r^n) =$$

$$= (f(\bigcup_{\alpha} X^{\alpha}, X^2, \ldots, X^n))_r = (\bigcup_{\alpha} f(X^{\alpha}, X^2, \ldots, X^n))_r =$$

$$= \bigvee_{\alpha} f(X^{\alpha}, X^2, \ldots, X^n)_r = \bigvee_{\alpha} \bar{f}(X_r^{\alpha}, X_r^2, \ldots, X_r^n).$$

Theorem 2. *Let $(A; F)$ be an algebra and $(A_r; F, \wedge, \vee)$ the lattice-ordered algebra of its r-ideals. If*

$$\varphi(x^1, \ldots, x^k) = \psi(x^1, \ldots, x^k)$$

is an identity in A such that

1. *φ and ψ are built up from r-admissible operations $f \in F$;*

2. *each of φ and ψ contains every variable x^i at most once explicitly,*[4])

then

$$\bar{\varphi}(X_r^1, \ldots, X_r^k) = \bar{\psi}(X_r^1, \ldots, X_r^k)$$

is an identity in A_r.

For the proof observe that in view of (3) and since no repetition of the x^i may occur in any member of the identity, $\bar{\varphi}(X_r^1, \ldots, X_r^k)$ is just the r-ideal generated by all $\varphi(x^1, \ldots, x^k)$ with $x^i \in X^i$. On account of the given identity, this is nothing else than the r-ideal generated by all $\psi(x^1, \ldots, x^k)$ with $x^i \in X^i$ which is in turn equal to $\bar{\psi}(X_r^1, \ldots, X_r^k)$. This completes the proof.

It is not always necessary to consider in $(A_r; F, \wedge, \vee)$ all the operations in F, it is sometimes sufficient to take only a subfamily of F as the family of fundamental operations in A_r. For instance, in the case of submodules we usually disregard in A_r from multiplications by ring elements and from subtraction.

The following problem is open: Given an algebra $(B; G, \wedge, \vee)$, when is it obtained from an algebra $(A; F)$ with $G \subseteq F$ as an r-ideal system? Note that B is always a complete lattice such that every element of B is the join of compact elements.[5]) It is also a hard problem to find conditions under which the isomorphy of r-ideal systems implies the isomorphy of the algebras from which they were constructed.

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[4]) Note that the commutative or associative identity satisfies condition 2, but distributivity does not.

[5]) Recall that an element c of a lattice K is said to be compact if $c \leqq \bigvee_{\alpha} x_{\alpha}$ for some (infinite) set $\{x_{\alpha}\} \subseteq K$ implies that $c \leqq x_{\alpha_1} \vee \ldots \vee x_{\alpha_t}$ for a finite number of the x_{α}.

2. The additive theory

If we regard the r-ideals of $(A; F)$ as analogues of the ideals in ring theory, then we may try to develop an additive ideal theory on the pattern of the Lasker—Noether theory of ideals in commutative rings. The fact that we have a lattice-ordered algebra $(A_r; F, \wedge, \vee)$ at our disposal which has been obtained from an algebra $(A; F)$ is not relevant in our discussions. Therefore we shall make use of the lattice-theoretic point of view developed in different generalizations of commutative ideal theory.

In what follows let $(L; F)$ be a lattice-ordered algebra [which will play the role of $(A_r; F, \wedge, \vee)$] satisfying the following conditions:

(i) the binary operations $\wedge, \vee$ are in F and under them L is a complete lattice;

(ii) every $f \in F$ is isotone in each of its variables with the whole of L as monotony domain;

(iii) every $f \in F$ is infinitely distributive over $\vee$, i. e.

$$f(x_1, \dots, \bigvee_\alpha x_i^\alpha, \dots, x_n) = \bigvee_\alpha f(x_1, \dots, x_i^\alpha, \dots, x_n)$$

for all x_j in L.

Note that our hypotheses imply that L contains a greatest and a least element, and if g is an operation composed of operations $f \in F$, then it is likewise infinitely distributive over $\vee$.

In addition to (i)—(iii) we also assume:

(iv) there is an operator Φ which associates with each element $x \in L$ an element $\Phi(x)$ of L such that

$$x \leqq \Phi(x).$$

In most cases it is useful to assume that Φ is a closure operator, i. e.

$$x \leqq \Phi(y) \text{ implies } \Phi(x) \leqq \Phi(y) \qquad \text{for } x, y \in L,$$

or that it is linear in the sense that

$$\Phi(x \wedge y) = \Phi(x) \wedge \Phi(y) \qquad \text{for all } x, y \in L.$$

With the aid of operator Φ one is able to introduce special types of elements corresponding to prime elements and generalizations in ring theory.[6])

Let f be an n-ary operation on L, $n \geqq 2$. We are going to call an element p of L an *(f, Φ)-prime* if

(a) for each $i = 1, \dots, n$ and for all $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in L$ we have

$$f(x_1, \dots, x_{i-1}, p, x_{i+1}, \dots, x_n) \leqq p,$$

(b) if f' is an operation built up from the operation f only,[7] and if

$$f'(x_1, \dots, x_k) \leqq p,$$

then

$$x_j \leqq \Phi(p) \qquad \text{for some } j = 1, \dots, k.$$

[6]) Examples for Φ may be found in [1], Chapter XII, section 6.

[7]) If e. g. f denotes multiplication, then f' is multiplication with several factors.

In particular, if Φ is the identity operator and if f stands for multiplication, then we have the just the notion of prime elements.

An element $q \in L$ is called (f, Φ)-*primary* if it satisfies, in addition to (a) (with p replaced by q), also

(c) if f' is as in (b) and if

$$f'(x_1, \ldots, x_k) \leqq q,$$

then for each i, $1 \leqq i \leqq k$,

$$\text{either } x_i \leqq q \text{ or } x_j \leqq \Phi(q) \text{ for some } j \neq i.$$

Clearly, this notion corresponds to primary ideals in commutative rings if $\Phi(x)$ is the radical of x. Observe that according to our definition, *every (f, Φ)-primary element is necessarily (f, Φ)-prime.*

We shall consider intersections $a = x_1 \wedge \ldots \wedge x_k$ of elements $a \in L$ by means of $x_i \in L$ possessing certain properties. We call such an intersection *irredundant* if no x_i may be omitted so as to obtain the same intersection a. It is called *short* if it is irredundant and no subset of the x_i has an intersection which has again the same property as the x_i under consideration.

The following results generalize those in ring theory and those in lattice-ordered semigroups.[8])

Theorem 3. *Let Φ be a closure operator. Then the intersection*

$$p = p_1 \wedge \ldots \wedge p_k$$

of a finite number of (f, Φ)-prime elements p_i is (f, Φ)-prime again if and only if

$$\Phi(p) = \Phi(p_j) \quad \textit{for some } j.$$

If p is (f, Φ)-prime, then take f' built up from the operation f only such that in the argument of f' each of p_i occurs at least once. Then by (a) and the isotony of f we have

$$f'(p_1, \ldots, p_k) \leqq p_i \quad \text{for each } i.$$

Hence $f'(p_1, \ldots, p_k) \leqq p$ which implies by assumption $p_j \leqq \Phi(p)$ for some $j = 1, \ldots, k$. Now Φ being a closure operator, $\Phi(p_j) \leqq \Phi(p)$. But $p \leqq p_j$ implies the converse inequality, thus $\Phi(p_j) = \Phi(p)$, establishing necessity. For the sufficiency, let $\Phi(p) = \Phi(p_j)$ for some j. Condition (a) is satisfied by p, because

$$f(x_1, \ldots, p, \ldots, x_n) \leqq f(x_1, \ldots, p_l, \ldots, x_n) \leqq p_l$$

for every l. If f' is an operation as stated in (b), and if $f'(x_1, \ldots, x_k) \leqq p$, then $f'(x_1, \ldots, x_k) \leqq p_j$ too which implies $x_l \leqq \Phi(p_j) = \Phi(p)$ for some index l.

Theorem 4. *If Φ is a linear closure operator and if*

$$a = p_1 \wedge \ldots \wedge p_k = p_1' \wedge \ldots \wedge p_m'$$

are two short decompositions of $a \in L$ into intersections of (f, Φ)-primes, then $k = m$, and the elements $\Phi(p_1), \ldots, \Phi(p_k)$ are the same as $\Phi(p_1'), \ldots, \Phi(p_m')$, up to order.

[8]) Cf. [1] Chapter XII, sections 4 and 5.

We begin the proof again with an f' as described in (b) such that $f'(p_1, \ldots, p_k)$ contains each p_j at least once. Then

$$f'(p_1, \ldots, p_k) \leqq p_1 \wedge \ldots \wedge p_k = a \leqq p'_j$$

implies $p_i \leqq \Phi(p'_j)$ for some $i = i(j)$. Therefore $\Phi(p_i) \leqq \Phi(p'_j)$. The same inference with p'_j replaced by p_i yields $\Phi(p'_l) \leqq \Phi(p_i)$ for some $l = l(i)$. Now $\Phi(p'_j \wedge p'_l) = \Phi(p'_j) \wedge \Phi(p'_l) = \Phi(p'_l)$ implies in view of the preceding theorem that $p'_j \wedge p'_l$ is again (f, Φ)-prime, consequently, $j = l$ and we have $\Phi(p_i) = \Phi(p'_j)$. The proof is completed.

Turning to (f, Φ)-primary elements, we have:

Theorem 5. *Let Φ be a closure operator. An irredundant intersection*

$$q = q_1 \wedge \ldots \wedge q_k$$

of a finite number of (f, Φ)-primary elements q_i is likewise (f, Φ)-primary if and only if

$$\Phi(q) = \Phi(q_j) \quad \textit{for each } j.$$

Let q be (f, Φ)-primary. Then we have by (a)

$$f(q_1 \wedge \ldots \wedge q_{j-1} \wedge q_{j+1} \wedge \ldots \wedge q_k, q_j, \ldots, q_j) \leqq q$$

for each j where the first argument of f is certainly not $\leqq q$ because of irredundancy. Thus $q_j \leqq \Phi(q)$ and hence $\Phi(q_j) \leqq \Phi(q)$. The converse inclusion also holds, hence $\Phi(q_j) = \Phi(q)$ for an arbitrary j. Conversely, let q satisfy $\Phi(q) = \Phi(q_j)$ for every j. Since (a) is obvious for q, assume $f'(x_1, \ldots, x_m) \leqq q$ for some f' as in (b). If $x_i \not\leqq q$, then $x_i \not\leqq q_j$ for some j. Therefore $f'(x_1, \ldots, x_m) \leqq q_j$ implies $x_l \leqq \Phi(q_j) = \Phi(q)$ for some $l \neq i$. Consequently, q is (f, Φ)-primary.

We shall need the following generalization of residuals. This is not the most general one which can be introduced here in the natural way, but this will suffice in our present case.

Given $a, b \in L$ and an n-ary operation f with $n \geqq 2$, let us consider the set of all $x \in L$ such that

$$f(x, b, \ldots, b) \leqq a.$$

If this set is not void, then by completeness and infinite distributivity it contains a maximum element which we shall denote by $a :_f b$. That is to say, $a :_f b$ satisfies:

$$x \leqq a :_f b \quad \text{if and only if} \quad f(x, b, \ldots, b) \leqq a.$$

It is readily seen that $\left(\bigwedge_\alpha a_\alpha\right) :_f b$ exists if and only if each of $a_\alpha :_f b$ exists and then

$$\left(\bigwedge_\alpha a_\alpha\right) :_f b = \bigwedge_\alpha (a_\alpha :_f b).$$

Theorem 6. *Let Φ be a linear closure operator and*

$$a = q_1 \wedge \ldots \wedge q_k = q'_1 \wedge \ldots \wedge q'_m$$

short decompositions of $a \in L$ into (f, Φ)-primary elements. Then $k = m$ and the elements $\Phi(q_1), \ldots, \Phi(q_k)$ are, up to order, equal to the elements $\Phi(q'_1), \ldots, \Phi(q'_m)$.

To begin with, observe that the elements $\Phi(q_1), \ldots, \Phi(q_k)$ are different, and so are the elements $\Phi(q_1'), \ldots, \Phi(q_m')$ by virtue of the assumption on Φ, Theorem 5 and shortness of decompositions. Pick out some maximal one amongst all $\Phi(q_i)$, $\Phi(q_j')$, say $\Phi(q_1)$, and form the residual

$$a:_f q_1 = q_1:_f q_1 \wedge \ldots \wedge q_k:_f q_1 .$$

This exists on account of (a). Since $q_1 \not\equiv \Phi(q_i)$ for $i>1$ [otherwise $\Phi(q_1) \leqq \Phi(q_i)$], from

$$f(q_i:_f q_1, q_1, \ldots, q_1) \leqq q_i$$

and from the (f, Φ)-primary character of q_i we infer $q_i:_f q_1 \leqq q_i$. On the other hand, by (a), $q_i:_f q_1 \geqq q_i$ whence $q_i:_f q_1 = q_1$ for $i>1$. Therefore[9])

$$a:_f q_1 = q_2 \wedge \ldots \wedge q_k$$

which is $>a$ by irredundancy. Hence

$$a:_f q_1 = q_1':_f q_1 \wedge \ldots \wedge q_m':_f q_1$$

implies $q':_f q_1 > q_j'$ for some j. From

$$f(q_j':_f q_1, q_1, \ldots, q_1) \leqq q_j'$$

we get $q_1 \leqq \Phi(q_j')$, and so $\Phi(q_1) \leqq \Phi(q_j')$. This proves that the maximal ones among $\Phi(q_i)$ and those among $\Phi(q_j')$ coincide.

Next let $\Phi(q_1) = \Phi(q_1')$ be maximal. Since by Theorem 5 $q_1 \wedge q_1'$ is again (f, Φ)-primary with $\Phi(q_1 \wedge q_1') = \Phi(q_1)$, we can replace q_1 and q_1' by their intersection in the representations of a; or, otherwise expressed, $q_1 = q_1'$ may be assumed. Then

$$a:_f q_1 = q_2 \wedge \ldots \wedge q_k = q_2' \wedge \ldots \wedge q_m',$$

and an obvious induction completes the proof.

It is to be emphasized that the existence of decompositions of the elements of L into the intersection of (f, Φ)-prime or -primary elements is not true in general. For the case in which f is multiplication we refer to [1] Chapter XII where several examples for Φ are exhibited.

Reference

[1] L. Fuchs, *Partially ordered algebraic systems* (Oxford—London—New York—Paris, 1963).

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[9]) From (a) it follows that $q_1:_f q_1$ is equal to the maximum element of L.